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FAST TRACK PAPER

Representations of the radiated energy in earthquakes

Luis Rivera[1] and Hiroo Kanamori[2]

[1]*Institut de Physique du Globe de Strasbourg (UMR CNRS-ULP 7516), 5, Rue René Descartes, F67084, Strasbourg, France. E-mail: luis@sismo.u-strasbg.fr*
[2]*Seismological Laboratory, California Institute of Technology, Pasadena, CA 91125, USA*

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SUMMARY

We investigate the representation of the radiated energy, E_R, in earthquakes. In seismology E_R is estimated from either far-field seismic waves or the stress and displacement on the fault plane. Although E_R comes from the entire volume of the Earth, it can be expressed as an integral over the fault plane. However, the integrand cannot be given a simple physical meaning such as the radiated energy density on the fault plane. The stress on the fault plane changes rapidly during a seismic rupture. Although the energy radiated by this process is not included in the estimate of E_R in a simplified practice in seismology, it is correctly included in the expression of E_R in standard seismological practice. Using the representation theorem, we can express E_R as a surface integral over the fault plane, with the integrand containing the slip function on the fault plane. However, the integrand at a point depends not only on the slip function at the point but also on the slip functions everywhere on the fault plane. Thus, the simple method in which E_R is estimated by summation of the local energy flux on the fault plane does not yield a correct estimate.

Key words: energy balance, fracture energy, friction energy, radiated energy, seismic energy, seismic source.

1 INTRODUCTION

The potential energy (mainly elastic strain energy and gravitational energy) stored inside the Earth is released during an earthquake and part of it is radiated as seismic waves (Kostrov 1974; Dahlen 1977). The radiated energy, E_R, is estimated in seismology from either far-field seismic waves (this has been done since the early days of seismology; Galitzin 1915; Jeffreys 1923) or the stress and displacement on the fault plane (Ide 2002; Favreau & Archuleta 2003). The radiated energy comes from the entire volume of the Earth, yet with either method it can be expressed as an integral over the fault plane and, because of this, it is sometimes implied that the radiated energy is distributed on the fault plane and radiated from there.

The stress on the fault plane changes rapidly during a seismic rupture. In a simplified practice in seismology, the energy radiated by this process is not included in the estimate of E_R, with the implication that E_R is underestimated in seismology. However, in standard seismological practice this energy is correctly included in the expression of E_R.

The far-field displacements can be uniquely determined by slip on the fault plane using the representation theorem. Then, E_R can be expressed as a surface integral over the fault plane, with the integrand containing the slip function on the fault plane. However, the integrand at a point depends not only on the slip function at the point but also on the slip functions everywhere on the fault plane. The different parts of the fault plane contribute to the far-field velocity either constructively or destructively, and the energy flux at far field which is proportional to the square of the velocity depends on the slip function over the entire fault plane. Thus, the simple method in which E_R is estimated by summation of the local energy flux on the fault plane (e.g. McGarr & Fletcher 2002) does not yield a correct estimate.

In view of occasional confusion about these issues in the literature, we address in this paper the physical meaning of the surface integral in the expression of E_R, the effect of rapid changes in stress on the fault plane on the seismological estimate of E_R and the difficulty in estimating E_R using a local energy flux on the fault plane.

2 RADIATED ENERGY

We recapitulate below some results concerning the energy budget of an earthquake without going into the details, which can be found elsewhere (e.g. Kostrov 1974; Dahlen 1977; Rudnicki & Freund 1981; Kostrov & Das 1988; Dahlen & Tromp 1998).

The radiated energy, E_R, is defined as the amount of energy that would be carried to the far field in the form of seismic waves if an earthquake occurred in an infinite and non-attenuating medium (Kostrov 1974). To estimate it, we take a closed surface S_0, embedded in the

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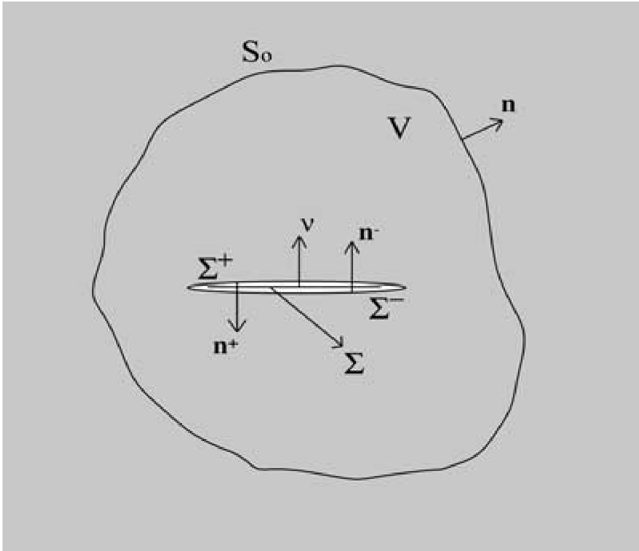

Figure 1. Schematic figure showing the surface S_0, volume V and the fault plane Σ surrounded by a closed surface made of Σ^+ and Σ^-.

medium and around the earthquake source at a distance, r, much larger than the source dimension, and calculate the work done by the stress perturbation across S_0 (Fig. 1) :

$$E_R \equiv -\int_{t_0}^{t_1} dt \int_{S_0} \left(\sigma_{ij} - \sigma_{ij}^0\right)\dot{u}_i n_j \, dS. \tag{1}$$

Here, σ_{ij} represents the stress at any time, σ_{ij}^0 is the initial stress, u_i is the displacement, $\dot{u}_i$ is the particle velocity and n_j is the unit vector normal to S_0 pointing outwards. t_0 is a reference time before the earthquake and t_1 is a time after the earthquake when any movement has ceased within S_0.

It can be shown (e.g. Rudnicki & Freund 1981) that (1), in the far field where it is defined, is equivalent to

$$E_R = \int_{t_0}^{t_1} \int_{S_0} \rho \left[\alpha(\dot{u}_j n_j)^2 + \beta(\dot{u}_i - \dot{u}_j n_j n_i)^2\right] dS\, dt \tag{2}$$

where ρ is the density and α and β are the P- and S-wave speeds. The '*radiated energy*' means the energy carried by seismic waves through S_0 which is embedded in the medium. This expression was used by Haskell (1964). In the case of a finite medium, like the Earth, we cannot define the radiated energy in this way. The elastic waves would remain trapped inside the Earth, bouncing back and forth on the free surface producing surface waves, reflected waves, etc. An alternative definition is '*seismic energy*' which is the total amount of energy dissipated due to anelasticity within the whole Earth after an earthquake (McCowan & Dziewonski 1977; Dahlen 1977; Dahlen & Tromp 1998).

Expression (2) has been used in seismology for decades to measure E_R (Galitzin 1915; Jeffreys 1923; Singh & Ordaz 1994; Choy & Boatwright 1995; Venkataraman & Kanamori 2004) by removing any free surface and attenuation effects from the data. An alternative approach (Vassiliou & Kanamori 1982; Kikuchi & Fukao 1988) is first to solve for the seismic source model of the earthquake, then to compute the elastic field generated by the same model within an infinite elastic medium at the far field and, finally, apply (1) or (2) to estimate E_R.

A completely different way to express E_R can be obtained from the energy budget (Kostrov 1974). We consider the same medium as above and model an earthquake as a stress release process on a surface (i.e. fault plane) Σ inside S_0. Let V be the volume inside S_0. The released potential energy (strain energy + gravitational energy + kinetic-rotational energy) in V associated with an earthquake can be estimated as follows.

Let us consider a pre-stressed elastic medium, undergoing an infinitesimal elastic deformation, $d\varepsilon_{ij}$, and let us ignore gravitation and rotation effects for the time being. The released elastic energy density is proportional to the total stress and is given by

$$\begin{aligned} dw &= -\sigma_{ij} d\varepsilon_{ij} \\ &= -\left(\sigma_{ij}^0 + c_{ijkl}\varepsilon_{kl}\right) d\varepsilon_{ij} \\ &= -\sigma_{ij}^0 d\varepsilon_{ij} - \frac{1}{2}\{c_{ijkl}\varepsilon_{kl} d\varepsilon_{ij} + c_{ijkl}\varepsilon_{ij} d\varepsilon_{kl}\} \\ &= -d\left\{\sigma_{ij}^0 \varepsilon_{ij} + \frac{1}{2} c_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\right\} \end{aligned} \tag{3}$$

where the strain ε_{ij} is measured from the initial state (i.e. $\varepsilon_{ij}^0 \equiv 0$).

Integrating between the initial and the final states we have

$$
\begin{aligned}
\Delta w &\equiv -\left\{\sigma_{ij}^0\varepsilon_{ij}^1 + \frac{1}{2}c_{ijkl}\varepsilon_{kl}^1\varepsilon_{ij}^1\right\} \\
&= -\left\{\sigma_{ij}^0\varepsilon_{ij}^1 + \frac{1}{2}(\sigma_{ij}^1 - \sigma_{ij}^0)\varepsilon_{ij}^1\right\} \\
&= -\frac{1}{2}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\varepsilon_{ij}^1 \\
&= -\frac{1}{2}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_{i,j}^1.
\end{aligned}
\tag{4}
$$

Using further the elastic equilibrium equations with zero body forces, both in the initial and final states, we have

$$\Delta w = -\frac{1}{2}\left\{(\sigma_{ij}^0 + \sigma_{ij}^1)u_i^1\right\}_{,j}. \tag{5}$$

Finally, integrating through the whole volume we can write the total released potential energy as

$$\Delta W = -\frac{1}{2}\int_V \left\{\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1\right\}_{,j} dV. \tag{6}$$

If gravitational and kinetic rotational effects are included in the analysis (Dahlen 1977) then (4) should be modified to include the corresponding energy contributions, but the equilibrium equations for the initial and final states should also be completed with the gravitational and centrifugal forces. It is remarkable that, by the end, eqs (5) and (6) remain unchanged and are then quite general including pre-stress, gravitation and kinetic-rotation effects in the energy balance.

The particular shape of (6) suggests the use of the Gauss theorem, leading to three surface integrals:

$$\Delta W = -\frac{1}{2}\int_V \left\{\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1\right\}_{,j} dV = -\frac{1}{2}\left\{\int_{S_o}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1 n_j dS + \int_{\Sigma^+}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1 n_j^+ dS + \int_{\Sigma^-}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1 n_j^- dS\right\} \tag{7}$$

where u_i^1 is the final displacement. This expression can be further simplified by defining $\Delta u_i = u_i^+ - u_i^-$, which is the displacement discontinuity from one side of the fault to the other, and letting Σ^+ and Σ^- collapse to a single open surface Σ, on which we choose a unit normal vector defined as $\nu_j \equiv n_j^- = -n_j^+$. With these definitions, the final expression for the total released potential energy within V becomes:

$$\Delta W = -\frac{1}{2}\int_V \left\{\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1\right\}_{,j} dV = -\frac{1}{2}\left\{\int_{S_o}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)u_i^1 n_j\, dS - \int_{\Sigma}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\Delta u_i \nu_j\, dS\right\} \tag{8}$$

On the other hand, Kostrov (1974) shows that part of the potential energy is expended on Σ and another part goes out through S_0, and derived the expression

$$\Delta W = \int_\Sigma 2\gamma_{\text{eff}}\, dS + \int_{t_0}^{t_1} dt \int_{\Sigma(t)} \sigma_{ij}\Delta\dot{u}_i \nu_j\, dS - \int_{t_0}^{t_1} dt \int_{S_0} \sigma_{ij}\dot{u}_i n_j\, dS \tag{9}$$

where $\Sigma(t)$ is the ruptured fault surface at time t, and γ_{eff} is the effective fracture energy. All the singular terms related to the stress concentration ahead of the crack tip have been collected in γ_{eff} (Kostrov 1974).

Equating (8) and (9), and using the definition of the radiated energy (1) leads to

$$E_{\text{R}} = \frac{1}{2}\int_\Sigma \left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\Delta u_i \nu_j\, dS - \int_\Sigma 2\gamma_{\text{eff}} dS - \int_{t_0}^{t_1} dt \int_{\Sigma(t)} \sigma_{ij}\Delta\dot{u}_i \nu_j dS + \frac{1}{2}\int_{S_0}\left(\sigma_{ij}^0 - \sigma_{ij}^1\right)u_i^1 n_j\, dS \tag{10}$$

The last term vanishes if S_0 is taken far enough from the fault, because u_i^1 and $\sigma_{ij}^0 - \sigma_{ij}^1$ decrease as $1/r^2$, and $1/r^3$, respectively. Then the final expression for E_{R} is given by

$$E_{\text{R}} = \frac{1}{2}\int_\Sigma \left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\Delta u_i \nu_j\, dS - \int_\Sigma 2\gamma_{\text{eff}}\, dS - \int_{t_0}^{t_1} dt \int_{\Sigma(t)} \sigma_{ij}\Delta\dot{u}_i \nu_j\, dS \tag{11}$$

as an integral of the expression that contains only the displacement and stress on the fault surface.

It may appear at first sight from (11) that E_{R} depends on the absolute value of the stresses, but a closer inspection shows that the pre-stress, σ_{ij}^0, is present both in the first and the third terms of the right-hand side of (11) and cancels out. Writing it explicitly, we have:

$$E_{\text{R}} = \frac{1}{2}\int_\Sigma \left(\sigma_{ij}^1 - \sigma_{ij}^0\right)\Delta u_i \nu_j dS - \int_\Sigma 2\gamma_{\text{eff}} dS - \int_{t_0}^{t_1} dt \int_{\Sigma(t)} \left(\sigma_{ij} - \sigma_{ij}^0\right)\Delta\dot{u}_i \nu_j\, dS. \tag{12}$$

An alternative expression is obtained after integrating (12) by parts:

$$E_{\text{R}} = \frac{1}{2}\int_\Sigma \left(\sigma_{ij}^0 - \sigma_{ij}^1\right)\Delta u_i \nu_j\, dS - \int_\Sigma 2\gamma_{\text{eff}} dS + \int_{t_0}^{t_1} dt \int_{\Sigma(t)} \dot{\sigma}_{ij}\Delta u_i \nu_j\, dS. \tag{13}$$

This is the same as eq. (2.26) of Kostrov (1974). Expressions (12) and (13) share two remarkable properties: first, they allow us to estimate E_{R} from the stresses and displacements only on the fault plane; second, it is not necessary to know the absolute value of stress on the fault. That is, E_R depends only on the stress perturbation on the fault caused by fracture and is independent of the pre-stress within the medium.

We then have two independent ways of estimating E_{R}, either with its original definition (1) or (2) which is given as an integral on S_0, or, indirectly, by using (12) or (13) as an integral over the fault surface Σ.

Eqs (12) and (13) above are derived for an infinite medium. However, exactly the same expressions can be obtained for a bounded medium like the Earth. In this case, S_0 is the free surface bounding the medium and, as discussed earlier, the energy attenuated in the body of the medium (i.e. seismic energy) is interpreted as $E_{\rm R}$ in eqs (12) and (13). Also, the first term of the right-hand side of (8) vanishes, and the first term of the right-hand side of (11) represents the change in the potential energy in the medium. This expression can be written in the form frequently used in seismology as

$$E_{\rm R} = \Delta W - E_{\rm G} - E_{\rm F} \tag{14}$$

where

$$\Delta W = \frac{1}{2}\int_\Sigma \left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\Delta u_i \nu_j \, dS$$

$$E_{\rm G} = \int_\Sigma 2\gamma_{\rm eff}\, dS$$

and

$$E_{\rm F} = \int_{t_0}^{t_1} dt \int_{\Sigma(t)} \sigma_{ij}\Delta\dot{u}_i \nu_j \, dS.$$

3 RADIATED ENERGY EXPRESSED AS A SURFACE INTEGRAL

As mentioned above, the term $\gamma_{\rm eff}$ in eqs (9)–(13) contains all the stress and velocity singularities related to rupture propagation, and the integral over $\Sigma(t)$ has no singularities. Under these circumstances, we can exchange the order of integration. If we take, for example, the last term in (11) we can write

$$\int_{t_0}^{t_1} dt \int_{\Sigma(t)} \sigma_{ij}\Delta\dot{u}_i\nu_j\, dS = \int_\Sigma dS \int_{t_r(\vec{\xi})}^{t_1} \sigma_{ij}\Delta\dot{u}_i\nu_j\, dt \equiv \int_\Sigma F_{\rm F}\, dS \tag{15}$$

where

$$F_{\rm F} = \int_{t_r(\vec{\xi})}^{t_1} \sigma_{ij}\Delta\dot{u}_i\nu_j\, dt \tag{16}$$

and $t_r(\vec{\xi})$ is the rupture starting time at $\vec{\xi} \in \Sigma$. We can now rewrite (11) as

$$E_{\rm R} = \int_\Sigma F_{\rm E}(\vec{\xi})\, dS - \int_\Sigma F_{\rm G}(\vec{\xi})\, dS - \int_\Sigma F_{\rm F}(\vec{\xi})\, dS = \int_\Sigma F_{\rm R}(\vec{\xi})\, dS \tag{17}$$

where

$$F_{\rm E} = \frac{1}{2}\left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\Delta u_i\nu_j, \qquad F_{\rm G} = 2\gamma_{\rm eff} \tag{18}$$

and

$$F_{\rm R}(\vec{\xi}) = F_{\rm E}(\vec{\xi}) - F_{\rm G}(\vec{\xi}) - F_{\rm F}(\vec{\xi}). \tag{19}$$

The function $F_{\rm R}(\vec{\xi})$ is the integrand of the radiated energy, $E_{\rm R}$, and is often interpreted as the 'radiated energy density' or 'seismic energy density' on the fault plane (Ide 2002; Favreau & Archuleta 2003). However, this interpretation does not accurately represent the physics involved. In eq. (17), $F_{\rm G}(\vec{\xi})$ and $F_{\rm F}(\vec{\xi})$ are, respectively, the local fracture and frictional energy, and represent the physical processes which are actually taking place on the fault; they can be computed point-wise for the fault element dS. In contrast, although $F_{\rm E}(\vec{\xi})$ is written in terms of stress and displacement on Σ, it does not represent the local process on the fault. When integrated over Σ it represents the change in the potential energy in the entire volume, as shown by (7) with the Gauss theorem. In other words, the often used energy balance relation such as (14) does not hold for the fault element dS. As a result, $F_{\rm R}(\vec{\xi})$ given by (19), though similar to (14), cannot be regarded as the energy released locally from dS.

This point can be more precisely stated as follows. We write the vector field $(\sigma_{ij}^0 + \sigma_{ij}^1)\Delta u_i^1/2$ in eq. (7) by F_j, i.e.

$$F_j \equiv \left(\sigma_{ij}^0 + \sigma_{ij}^1\right)\Delta u_i^1/2 \tag{20}$$

which is defined at every point in V. We define its 'integral lines' by connecting the tangents to F_j at every point in V. Only one integral line passes through every point in V. (For example, if F_j is an electric field, then the integral lines are the lines of force; Stratton 1941, p. 161). We take a surface element dS on the fault plane, and consider the integral lines Γ passing through dS. Let V_Γ be the tubular volume defined by Γs coming from dS, and S_Γ be the cross sectional area normal to Γ (Fig. 2). Then,

$$\int_{V_\Gamma} F_{j,j}\, dV = \int_{S_\Gamma}\int_\Gamma F_{j,j}\, d\Gamma\, dS = F_j\nu_j\, dS|_{\text{on } S_0} - F_j n_j\, dS|_{\text{on }\Sigma} = -F_E(\vec{\xi})\, dS|_{\text{on }\Sigma} \tag{21}$$

where the property that Γ is tangent to F_j is used. The term $F_j n_j\, dS|_{\text{on } S_0}$ vanishes because S_0 is the free surface. In other words, (21) is the result of applying the Gauss theorem to the tubular volume V_Γ. Since $F_{j,j}$ is the volume density of the released energy, eq. (21) means that $F_{\rm E}(\vec{\xi})\, dS$ represents the total energy released from the tubular volume coming from dS, not the energy released from the surface element $dS|_{\text{on }\Sigma}$.

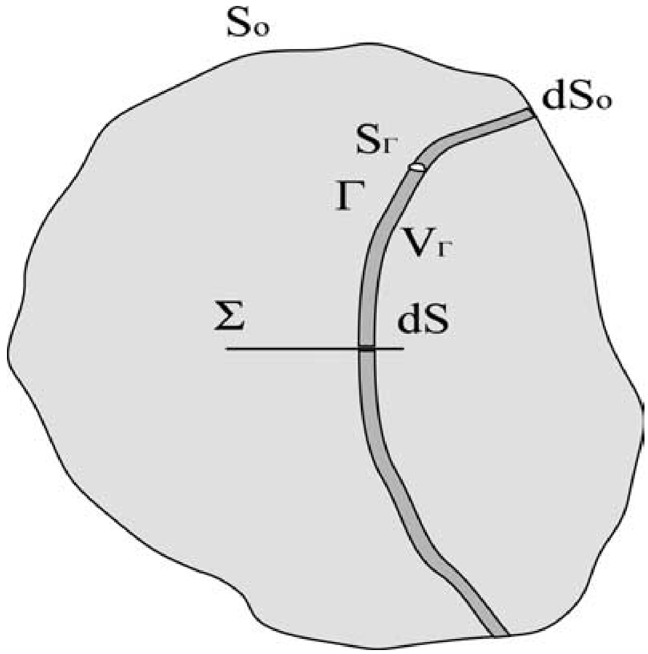

Figure 2. The tubular volume formed by integral lines passing through a surface element *dS*.

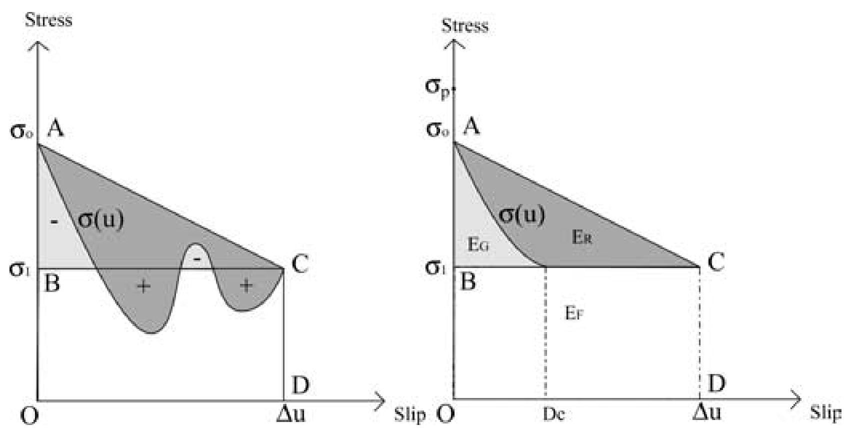

Figure 3. Graphical representation of the energy budget: (a) general case, (b) slip-weakening model.

4 SIMPLE MODEL

The interpretation of the energy budget can be facilitated using a simple model illustrated in the following. Combining (11) and (15), we obtain

$$E_{\mathrm{R}} = \frac{1}{2}\int_{\Sigma} \left(\sigma_{ij}^{0} + \sigma_{ij}^{1}\right) \Delta u_i \nu_j \, dS - \int_{\Sigma} 2\gamma_{\mathrm{eff}} \, dS - \int_{\Sigma} dS \int_{t_r(x)}^{t_1} \sigma_{ij} \Delta \dot{u}_i \nu_j \, dt. \tag{22}$$

If we consider a simple shear fault for which σ_{ij}^0, σ_{ij}^1 and Δu_i are uniform on the fault with area S and given by scalars σ^0, σ^1 and Δu, respectively, (22) can be written as

$$E_{\mathrm{R}} = \frac{1}{2}(\sigma^0 + \sigma^1)\Delta u S - \int_{\Sigma} 2\gamma_{\mathrm{eff}} \, dS - S\int_0^{\Delta u} \sigma \, d(\Delta u). \tag{23}$$

The second term on the right-hand side of (23) is the fracture energy E_{G}. For the time being, we ignore this term. The first term is the total change in the potential energy and the third term, the frictional energy. Fig. 3(a) shows a graphic representation of these energies per unit area. The potential energy change is given by the trapezoidal area AODC, and the frictional energy is given by the area under the curve labelled as σ (u). Thus, the radiated energy E_{R} is given by the dark area.

Eq. (23) can be also written as

$$E_{\mathrm{R}} = \frac{1}{2}(\sigma^0 - \sigma^1)\Delta u S - \int_{\Sigma_1} 2\gamma_{\mathrm{eff}} \, dS - \left(S\int_0^{\Delta u} \sigma \, d(\Delta u) - \sigma^1 \Delta u S\right). \tag{24}$$

The first term corresponds to the triangular area ABC, and the last term in the bracket gives the difference between the areas labelled by + and −, which Kostrov (1974) called the 'radiation friction'. In some simplified seismological practice, the triangular area, $\frac{1}{2}(\sigma^0 - \sigma^1)\Delta u S$, is taken as the radiated energy. Thus, in such a practice, as Kostrov (1974) pointed out, the term corresponding to the radiation friction is ignored in the estimation of E_{R}. However, in many other modern practices, E_{R} is measured from either far-field displacements using (2), or integration of (12) on the fault plane (Ide 2002; Favreau & Archuleta 2003). Thus, in principle, the term corresponding to the radiation friction is correctly included in estimation of E_{R}, though, in practice, it is always difficult to accurately include the contributions from high-frequency seismic waves. This point is often confused in the literature (e.g. Husseini 1977).

In the widely used slip-weakening model (e.g. Li 1987), the stress increases from σ^0 to a peak stress σ^{P} and drops to σ^1 over slip D_{c}, then the slip continues at a constant stress $\sigma = \sigma_1$ (Fig. 3b). For simplicity, here we ignore the difference between σ^{P} and σ^0. In this model, the grey area is interpreted as the fracture energy, E_{G}, and the rectangular area BODC is interpreted as the frictional energy, E_{F}. Usually, the grey area implicitly includes the second term on the right-hand side of (22) or (23) through the slip-weakening interpretation of energy dissipation in the crack-tip breakdown zone (e.g. Li 1987).

5 THE RADIATED ENERGY AND THE DISPLACEMENT HISTORY ON THE FAULT (DISLOCATION)

The representation theorem (de Hoop 1958; Burridge & Knopoff 1964) provides an expression for the displacement field of a finite fault as an integral on the fault surface. The integrand is particularly simple for the far field, and the expression for E_{R} can be obtained if slip $D(\vec{\xi}, t)$ is given on the fault as a function of position $\vec{\xi}$ on the fault plane (i.e. $\vec{\xi} \in \Sigma$). In fact, the energy flowing out at every moment at any point on S_0 is determined by interaction of the displacement due to all the patches on the fault. This result is valid regardless of the degree of complexity of fault geometry and the slip time history. Here, for simplicity, we restrict ourselves to the case in which the fault strike, the slip direction and the rupture direction do not vary on the fault. In this case, the far-field particle velocity components at $\vec{x}$ are given in a spherical coordinate system (r, θ, ϕ) with the origin at a corner of the fault (Haskell 1964) as:

$$\begin{aligned}
\dot{u}_r(\vec{x}, t) &= \left(\frac{\beta}{\alpha}\right)^3 \frac{R_P}{4\pi\beta r} I^{\alpha}(\vec{x}, t) \\
\dot{u}_\theta(\vec{x}, t) &= \frac{R_{S\theta}}{4\pi\beta r} I^{\beta}(\vec{x}, t) \\
\dot{u}_\phi(\vec{x}, t) &= \frac{R_{S\phi}}{4\pi\beta r} I^{\beta}(\vec{x}, t)
\end{aligned} \tag{25}$$

where R_P, $R_{S\theta}$ and $R_{S\phi}$ are the radiation patterns. In the following, c stands for either α or β. I^c is given by:

$$I^c(\vec{x}, t) = \int_\Sigma \ddot{D}\left(\vec{\xi}, t - \frac{|\vec{x} - \vec{\xi}|}{c}\right) d\vec{\xi}. \tag{26}$$

Substituting (25) and (26) into (2), we obtain

$$E_{\mathrm{R}} = \frac{\rho}{16\pi^2\beta} \int_{S_0} \int_{t_0}^{t_1} \left[\left(\frac{\beta}{\alpha}\right)^5 R_P^2(\theta, \phi) F_\alpha(\vec{x}, t) + R_S^2(\theta, \phi) F_\beta(\vec{x}, t) \right] dt d\Omega \tag{27}$$

where $R_S^2 \equiv R_{S\theta}^2 + R_{S\phi}^2$, $d\Omega$ is the solid angle dS/r^2, and

$$F_c(\vec{x}, t) = (I^c)^2 = \iint_{\Sigma\times\Sigma} \ddot{D}\left(\vec{\xi}, t - \frac{|\vec{x} - \vec{\xi}|}{c}\right) \ddot{D}\left(\vec{\eta}, t - \frac{|\vec{x} - \vec{\eta}|}{c}\right) d\vec{\xi} d\vec{\eta}. \tag{28}$$

In the above, the double integration is over the same fault plane, Σ. $F_c(\vec{x}, t)$ is proportional to the energy flux at $\vec{x} \in S_0$ and t. The integrand of (28) gives the 'contribution' from a couple of patches on the fault at $\vec{\xi}$ and $\vec{\eta}$ to the energy propagating through S_0 at $(\vec{x}, t)$. It depends on the way the displacements at $\vec{x} \in S_0$, generated by a couple of patches at $\vec{\xi}$ and $\vec{\eta}$, interact with each other. The patches can contribute positively or negatively to the final amplitude and to the energy flux, depending on whether they interact constructively or destructively. In this sense, the integrand of (28) can be regarded as 'radiated energy density' on $\Sigma \times \Sigma$. For example, we can explicitly write the S wave part of (27), after changing the order of integration, as

$$\begin{aligned}
E_{\mathrm{R}}^S &= \frac{\rho}{16\pi^2\beta} \iint_{\Sigma\times\Sigma} \int_{S_0} R_S^2(\theta, \phi) \int_{t_0}^{t_1} \ddot{D}\left(\vec{\xi}, t - \frac{|\vec{x} - \vec{\xi}|}{\beta}\right) \ddot{D}\left(\vec{\eta}, t - \frac{|\vec{x} - \vec{\eta}|}{\beta}\right) dt\, d\Omega\, d\vec{\xi}\, d\vec{\eta} \\
&= \iint_{\Sigma\times\Sigma} F_{\Sigma\Sigma}(\vec{\xi}, \vec{\eta})\, d\vec{\xi}\, d\vec{\eta}
\end{aligned} \tag{29}$$

which can be reduced to a form similar to (17):

$$\begin{aligned}
E_{\mathrm{R}}^S &= \frac{\rho}{16\pi^2\beta} \int_\Sigma \left\{ \int_{S_0} R_S^2(\theta, \phi) \int_{t_0}^{t_1} \ddot{D}\left(\vec{\xi}, t - \frac{|\vec{x} - \vec{\xi}|}{\beta}\right) \left[\int_\Sigma \ddot{D}\left(\vec{\eta}, t - \frac{|\vec{x} - \vec{\eta}|}{\beta}\right) d\vec{\eta} \right] dt\, d\Omega \right\} d\vec{\xi} \\
&= \int_\Sigma F_\Sigma(\vec{\xi})\, d\vec{\xi}
\end{aligned} \tag{30}$$

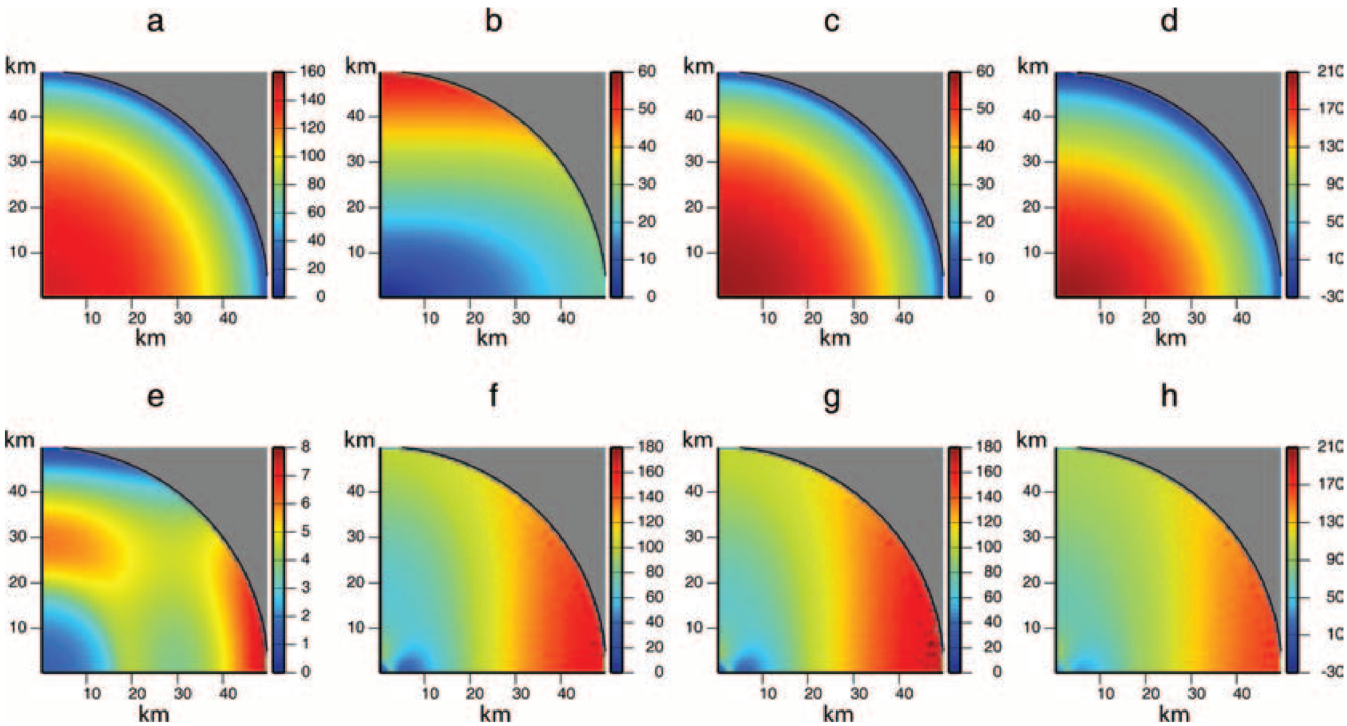

Figure 4. Integrands of the radiated energy E_{R} given by the two representations (17) and (30) for the Sato & Hirasawa (1973) fault model with a rupture velocity of 0.9β and a radius of 50 km. Only a quadrant of the fault is represented. Values in the colour scale are given in units of 10^{19} J km^{-2}. Top: (a) $F_{\mathrm{E}}(\vec{\xi}) - \sigma^0_{ij}\Delta u_i \nu_j$; (b) $F_{\mathrm{G}}(\vec{\xi})$; (c) $-(F_{\mathrm{F}}(\vec{\xi}) - \sigma^0_{ij}\Delta u_i \nu_j)$ and (d) $F_{\mathrm{R}}(\vec{\xi})$. Bottom: (e) $F_{\Sigma}(\vec{\xi})$ for *P* waves; (f) $F_{\Sigma}(\vec{\xi})$ for *S* waves; (g) sum of $F_{\Sigma}(\vec{\xi})$ for *P* and *S* waves; (h) the same as (g) shown with the same scale as (d). Note the very different patterns between (d) and (h); yet, when integrated, they give the same E_{R}.

The main difference between (29) and (30) is that (29) is a point-wise relationship; $F_{\Sigma\Sigma}(\vec{\xi}, \vec{\eta})$ relates in detail each couple of patches on Σ to the flux at each point on S_0. In contrast, $F_{\Sigma}(\vec{\xi})$ in (30) contains the contribution of the patch at $\vec{\xi}$, which interacts with the whole fault. In other words, $F_{\Sigma}(\vec{\xi})$ depends not only on the slip history at $\vec{\xi}$ but also on the slip distribution on the entire fault $D(\vec{\eta}, t)$.

Thus, the simple method used in several studies (e.g. McGarr & Fletcher 2002) in which the radiated energy is estimated by summation of the local energy flux on the fault plane does not yield a correct estimate.

Fig. 4 (bottom) shows the integrand $F_{\Sigma}(\vec{\xi})$ for the circular fault model of Sato & Hirasawa (1973). Since $F_{\Sigma}(\vec{\xi})$ is symmetrical with respect to both the $\phi = 0$ (the slip direction) and $\phi = 90^\circ$ diameters, $F_{\Sigma}(\vec{\xi})$ is shown only for a quadrant of the fault. For comparison, Fig. 4 (top) shows the integrands $F_{\mathrm{E}}(\vec{\xi}) - \sigma^0_{ij}\Delta u_i \nu_j$, $F_{\mathrm{G}}(\vec{\xi})$, $-(F_{\mathrm{F}}(\vec{\xi}) - \sigma^0_{ij}\Delta u_i \nu_j)$ and $F_{\mathrm{R}}(\vec{\xi})$ computed with eqs (16), (18) and (19). This computation is the same as that made by Ide (2002). Note that $F_{\mathrm{R}}(\vec{\xi})$ (Fig. 4d) and $F_{\Sigma}(\vec{\xi})$ (Fig. 4h) are very different, yet the integrals of these integrands over the entire fault plane yield exactly the same E_{R}. This example illustrates that the integrand of the radiated energy is not unique and cannot be given a simple physical meaning like the radiated energy density.

Fig. 5 shows $F_{\Sigma}(\vec{\xi})$ for the Haskell (1964) model. For this computation, fault length $L = 100$ km, fault width $W = 5$ km, dislocation $D = 1$ m, rise-time of dislocation $T = 3.0$ s and rupture speed $V = 3.58$ km s^{-1} are used. The *P*-wave speed, *S*-wave speed and the density of the medium are 8.0 km s^{-1}, 4.62 km s^{-1} and 3.0 g cm^{-3}, respectively. To avoid the singularity in $\ddot{D}$, the time function is smoothed over 0.1 s. Note that $F_{\Sigma}(\vec{\xi})$ vanishes in the middle section of the fault, which is the result of the interaction of a pair of patches at $\vec{\xi}$ and $\vec{\eta}$ discussed earlier.

6 CONCLUSION

The radiated energy, E_{R}, in earthquakes can be represented by a surface integral on the fault plane Σ. As shown by (11), the integrand of the surface integral contains the term $\frac{1}{2}(\sigma^0_{ij} + \sigma^1_{ij})\Delta u_i \nu_j$. This term cannot be interpreted as the local energy density at a point $\vec{\xi}(\vec{\xi} \in \Sigma)$ on the fault plane. We can show that it represents the energy released from the tubular volume formed by the integral lines of the vector $F_j \equiv \frac{1}{2}(\sigma^0_{ij} + \sigma^1_{ij})\Delta u_i$ passing through a unit area at $\vec{\xi}$ on the fault plane. Thus, this represents the energy coming from the volume of the medium, rather than the fault plane.

The stress on the fault plane changes rapidly during a seismic rupture. In a simplified practice in seismology in which the radiated energy is estimated by $E_{\mathrm{R}} = \frac{1}{2}(\sigma^0 - \sigma^1)\Delta u S$, the energy radiated by this process is not included in the estimate of E_{R}, with the implication that E_{R} is underestimated in seismology. However, in standard seismological practice, this energy is correctly included in the expression of E_{R}.

A simple method in which E_{R} is estimated by summation of the local energy flux on the fault plane is often used in seismology. Using the representation theorem we showed that E_{R} can be expressed as a surface integral over the fault plane, with the integrand containing the

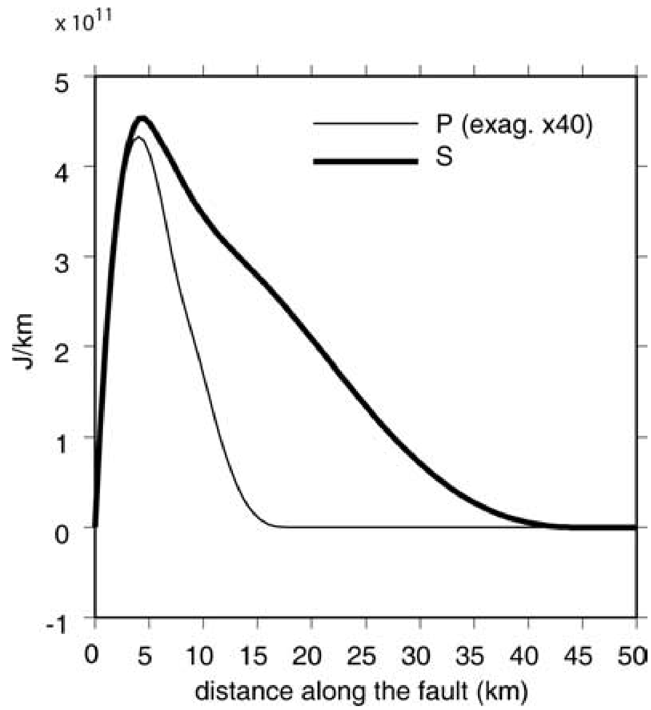

Figure 5. The integrand $F_\Sigma(\vec{\xi})$ for the Haskell (1964) model. Only one half of the total fault length is shown.

slip function on the fault plane. However, the integrand at a point depends on not only the slip function at the point, but also the slip functions everywhere on the fault plane. Thus, the simple method using the summation of energy flux computed for each point on the fault plane does not yield a correct estimate for E_R.

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